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DE FINETTI REPRESENTATIONS OF SURVIVAL FUNCTIONS LEVEL TO A PRODUCT MEASURE

Interim Technical Report

by

Dr. Roger M. Cooke

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de Finetti Representations of Survival Functions Level to a Product Measure

Roger M. Cooke *

June 18, 1992

Abstract

de Finetti-type representations of survival functions are considered. Exchangeable continuous strictly monotonic infinite-dimensional survival functions whose finite dimensional marginals have the same level sets as a product survival function, can be represented uniquely as mixtures of positive powers of that product survival function. A functional equation characterizes the survival functions which can be represented in this way, and the product measure is extracted using techniques from functional equations.

Key Words: l_p isotropic measures, de Finetti theorem, Schröder equation, bisymmetry.

1 Introduction

Two functions, f and g on $\Re^n$, with $n \geq 2$ are called *level* ($f \sim g$) if they have the same level sets, that is, $f(x) = f(y)$ if and only if $g(x) = g(y)$. This

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concept is meaningful for measures only if the measures are represented as integrals of density functions. It is convenient to study survival functions, that is functions of the form $\bar{F}(x) = \text{Prob}\{X > x\} = \text{Prob}\{X_1 > x_1, \cdots X_n > x_n\}$ for a random vector $X \in \Re_+$. We shall be concerned with continuous survival functions which are strictly monotone in each argument. For univariate survival functions this entails that an inverse exists; for n-dimensional multivariate survival functions, it entails that the level sets are simple $n-1$ dimensional hypersurfaces intersecting all the coordinate axes. Each point of the support is on exactly one such hypersurface.

This paper studies de Finetti-type representations of continuous strictly monotone survival functions which are level to a product measure. Let $\Re_+$ denote the positive reals, and $\bar{\Re}_+$ the non-negative reals. If $\bar{F} : \bar{\Re}_+^\infty \to [0,1]$ is a survival function and $x \in \bar{\Re}_+^n$, then $\bar{F}(x)$ denotes the n-dimensional marginal $\bar{F}(x_1, \ldots, x_n, 0, 0, \ldots)$, and when the context would be ambiguous, $\bar{F}_n$ will denote the n-dimensional marginal. $\bar{F} \sim \prod \mu$ means that all finite-dimensional marginals of the two functions $\bar{F}$ and $\prod \mu$ are level. Recall that according to the de Finetti theorem, any exchangeable survival function $\bar{F}; \bar{\Re}_+^\infty \to [0,1]$ can be written as

$$\forall x \in \Re^n; \bar{F}(x) = \int_{\mu \in \mathcal{M}} \prod_{i=1}^{n} \mu(x_i) d\lambda_{\bar{F}}\mu \tag{1}$$

where $\mathcal{M}$ is the set of survival measures on $[0, \infty)$. The measure $\lambda_{\bar{F}}$ is unique. In some cases the integral can be expressed as an integral over the range of some real parameter. For example, if we require that the density f of $\bar{F}$ is l_p- *isotropic*, that is, has level sets corresponding to the l_p norm, then the k-dimensional marginal density f_k can be uniquely written as ([4], [9], [11])

$$f_k\left(\sum_{i=1}^{k} x_i^p\right) = \int_0^\infty \exp\left[-t\sum_{i=1}^{k} x_i^p\right]\left[\frac{pt^{1/p}}{\Gamma(1/p)}\right]^k \lambda_{\bar{F}}(dt). \tag{2}$$

Comparing the above two representations, two differences are apparent:

(i) in the second equation the integral runs over values of a "scale parameter" t, and, (ii) the product measures over which the measure $\lambda_{\bar{F}}$ mixes are level to the density on the left hand side. These two facts are equivalent for continuous monotone survival functions, in fact:

Theorem 1 *Let $\bar{F} : \bar{\Re}_{+}^{\infty} \to [0,1]$ be a continuous strictly monotone survival function and μ be a continuous strictly monotone univariate survival function, then the following are equivalent:*

$$\bar{F} \sim \prod \mu \tag{3}$$

$$\forall n \in N, \forall x \in \bar{\Re}_{+}^{n};\ \bar{F}(x) = \int_{0}^{\infty} \prod_{i=1}^{n} \mu(x_i)^s \lambda_{\bar{F}}(ds) \tag{4}$$

It is trivial that (3) follows from (4). l_1-isotropic survival functions have l_1-isotropic densities, and converseley (this is not true for l_p densities with $p \neq 1$) [6]. The reverse implication then follows from (2) upon making the substitution $z(x) = -\ln(\mu(x))$. The family of survival functions $e^{-sz}, s \in (0, \infty)$ are said to have proportional hazard functions.

There are advantages to looking at the problem from the viewpoint of Theorem 1 In particular it suggests techniques of functional equations for finding equivalent conditions for (3), and for finding μ when these conditions are satisfied. For exchangeable $\bar{F}$, equivalent conditions in terms of a bisymmetry condition on the two-dimensional marginals of $\bar{F}$ are derived in Section 2. If X is exponentially distributed and $f(\cdot)$ is non-negative, continuous and increasing with $f(0) = 0$, then Mendel and Barlow [10] call the distribution of $f(X)$ "generalized Weibull". Section 2 shows how to recognize mixtures of generalized Weibulls from their 2-dimensional marginals, and how to extract the transformation $f = -\ln(\mu)$. This is applied in Section 3 to obtain a representation of l_p isotropic survival functions. Section 4 gives two functional equations for survival functions. The last of these could be derived from (2), but a direct proof using techniques of functional equations, enables us to give a short proof of Theorem 1 in the last Section.

2 Equivalent conditions for $\bar{F} \sim \prod \mu$

Throughout this section we assume that $\bar{F}$ is a strictly monotonic exchangeable survival function on $\Re_+^\infty$, and that μ is a continuous strictly monotonic univariate survival function. The first theorem collects some obvious properties, the second shows that it suffices to look at 2-dimensional marginals, and the third gives a functional equation which represents all $\bar{F}$ which are level to a product.

Theorem 2 *(1) If $\bar{F} \sim \prod \mu$ and $\mu(x) = \bar{F}(x)$ (i.e. μ is the 1-dimensional marginal of $\bar{F}$), then $\bar{F} = \prod \mu$.*
(2) If $\bar{F} \sim \prod \mu$ then

$$\bar{F}(x_{n+1}|x_1,\cdots x_n) = \bar{F}(x_{n+1}|y_1,\cdots y_n) \Longleftrightarrow \bar{F}(x_1,\cdots x_n) = \bar{F}(y_1,\cdots y_n) \tag{5}$$

Proof: Immediate.

Theorem 3 *Suppose for all $x, x' \in \Re_+^2$: $\bar{F}_2(x) = \bar{F}_2(x')$ if and only if $\mu(x_1)\mu(x_2) = \mu(x_1')\mu(x_2')$; then $\bar{F} \sim \prod \mu$.*

Proof: Put $z_i(x_i) = -\ln(\mu(x_i))$ and $\tilde{F}(z_1,\cdots,z_n) = \bar{F}(x_1,\cdots,x_n)$.

Lemma: A function $H : \Re^\infty \to \Re$ which is invariant under finite permutations and l_1 isotropic in the first two coordinates, is l_1 isotropic.

Proof of lemma: We must show for any $x_1,\cdots x_n$, $H(x_1,\cdots x_n) = H(\sum_{i=1}^n x_i, \overbrace{0,\cdots,0}^{n-1})$.

$$H(\sum_{i=1}^n x_i, \overbrace{0,\cdots,0}^{n-1}) = H(x_1, \sum_{i=2}^n x_i, \overbrace{0,\cdots,0}^{n-2}) =$$

$$H(0, \sum_{i=2}^n x_i, x_1, \overbrace{0,\cdots,0}^{n-3}) = H(x_2, \sum_{i=3}^n x_i, x_1, \overbrace{0,\cdots,0}^{n-3}) =$$

$$\cdots H(x_n, \cdots, x_1) = H(x_1, \cdots, x_n).\square$$

By the hypothesis of the theorem, if $z_1 + z_2 = z_1' + z_2'$, then $\tilde{F}(z_1, z_2) = \tilde{F}(z_1', z_2')$, i.e. $\tilde{F}$ is l_1 isotropic in the first two coordinates. The lemma entails that $\tilde{F}$ is l_1 isotropic, hence for x, $x' \in \bar{\Re}_+^n$, $z(x) = (\ln \mu(x_1), \cdots, \ln \mu(x_n))$; we have: $\bar{F}(x) = \bar{F}(x') \Leftrightarrow \tilde{F}(z) = \tilde{F}(z') \Leftrightarrow \sum_{i=1}^n z_i = \sum_{i=1}^n z_i' \Leftrightarrow \prod_{i=1}^n \mu(x_i) = \prod_{i=1}^n \mu(x_i')$. It follows that $\bar{F} \sim \prod \mu.\square$

For the next theorem we define $G : \bar{\Re}_+^2 \to \bar{\Re}_+$ as:

$$G(x, y) = z \text{ if } \bar{F}(z, z) = \bar{F}(x, y).$$

G is called *bisymmetric* if

$$G(G(x, y), G(z, w)) = G(G(x, z), G(y, w)).$$

The problem of identifying those survival functions which are level to a product is essentially a matter of representing bisymmetric forms.

Theorem 4 *$\bar{F}$ is level to a product if and only if G is bisymmetric.*

Proof: By Theorem 3 it suffices to find a univariate survival function μ such that for all $x, x', y, y' \in \bar{\Re}_+$;

$$\mu(x)\mu(y) = \mu(x')\mu(y') \text{ if and only if } \bar{F}(x, y) = \bar{F}(x', y').$$

The function G is reflexive and symmetric. There exists a continuous strictly monotonic function $k : \bar{\Re}_+ \to \Re$ such that

$$G(x, y) = k^{-1}\left(\frac{k(x) + k(y)}{2}\right);\ x, y \in \bar{\Re}_+. \tag{6}$$

Moreover, k is uniquely determined up to an affine transformation [2, p.287, p 246]. Arranging that $k(0) = 0, k(x) \geq 0$, put $\mu(x) = e^{-k(x)}$, then

$$\mu(G(x,y))^2 = \mu(x)\mu(y).$$

If $\bar{F}(x,y) = \bar{F}(x',y')$, then $G(x,y) = G(x',y')$ and $\mu(x)\mu(y) = \mu(x)\mu(y)$. $\square$

From the proof of [2, p.87], one can construct the function k and hence μ. However, if the conditions of Theorem 4 hold it is easier to recover μ in a different way. Let $\bar{F}_1$ denote the one-dimensional marginal of $\bar{F}$, and define

$$g_k(x) = \bar{F}_1^{-1}(\bar{F}(\overbrace{x, x, \cdots, x}^{k})$$

Then $\mu(x)^k = \mu(g_k(x)))$, or more generally,

$$\mu(x)^{k/n} = \mu(g_n^{-1}(g_k(x))),\ k, n \in N. \tag{7}$$

Hence, if we assign $\mu(x_0) = r$, for some $0 < r < 1$, then (7) determines μ^{-1} on a dense set. Since μ is continuous, μ is determined on a dense set as well. In specific situations more elegant methods may be available, as illustrated in the next section.

3 Applications to linear, inverse linear and l-p isotropic survival functions

For a uniquely defined continuous function $g : \bar{\Re}^+ \to \bar{\Re}^+$ we may write

$$\bar{F}(g(x), g(x)) = \bar{F}(0, x) \tag{8}$$

If $\bar{F} \sim \prod \mu$, then by Theorem 1 μ also satisfies (8), or

$$\log(\mu(g(x)) = \frac{1}{2}\log(\mu(x)) \tag{9}$$

Equation (9) is an example of the Schröder equation, and has been studied extensively [8]. Existence and uniqueness theorems are available for certain g, as illustrated by ([8, pp 68,69], [7]):

Theorem 5 *Let g be continuous and strictly increasing on $D = [0, a]$, $0 < a \leq \infty$; such that (i) $0 < g(x) < x$ in $D\backslash\{0\}$, (ii) $x \rightarrow g(x)/x$ is monotonic in $D\backslash\{0\}$, and (iii) $\lim_{x\rightarrow 0}[g(x)/x] = s$, where $0 < s < 1$. Then $\phi(g(x)) = s\phi(x)$ has a unique one parameter family of solutions $\phi : D \rightarrow \Re$ given by $\phi(x) = c\lim_{n\rightarrow\infty} \frac{g^n(x)}{g^n(x_0)}$. The function $x \rightarrow \frac{\phi(x)}{x}$ is monotonic in $D\backslash\{0\}$. where $c \in \Re$ is any constant and x_0 is an arbitrary fixed point in $D\backslash\{0\}$.*

This result can be applied to represent survival functions level to infinite products of *linear* $(1 - x;\ 0 \leq x \leq 1)$ and *inverse linear* $(\frac{1}{1+x};\ 0 \leq x, \infty)$ survival functions respectively. These correspond to uniform and quadratic densities.

Theorem 6 *Let $\bar{F}$: $D^\infty \rightarrow [0,1]$ be a continuous monotone exchangeable survival function and suppose that $\bar{F} \sim \prod \mu$.*

(i) If $D = [0,1)^\infty$ and $\bar{F}(1-(1-x)^{1/2}, 1-(1-x)^{1/2}) = \bar{F}(0,x)$, then $\mu(x) = 1 - x$.

(ii) If $D = [0,\infty)^\infty$ and $\bar{F}((1+x)^{1/2} - 1, (1+x)^{1/2} - 1) = \bar{F}(0,x)$, then $\mu(x) = \frac{1}{1+x}$.

Proof: We do only (ii) as (i) is similar. The function $g(x) = (1+x)^{1/2} - 1$ in (9) is easily verified to satisfy conditions (i) and (ii) of Theorem 5, and with L'Hospital's rule $\lim_{x\rightarrow 0} \frac{g(x)}{x} = \lim_{x\rightarrow 0} \frac{dg(x)}{dx} = 1/2$. It follows that a solution exists and is unique up to a constant. $\mu(x) = \frac{1}{1+x}$ is a solution and is therefore the only solution in the class of univariate survival functions. □

Results like Theorem 5 place restrictions on g which are not satisfied for general polynomial and inverse polynomial survival functions. A characterization of survival functions level to products of polynomial and inverse polynomial survival functions is not known. l_p isotropic survival functions violate condition (iii). A representation for l_p isotropic survival functions is given in:

Theorem 7 *Let $\bar{F}$: $[0,\infty)^\infty \rightarrow [0,1]$ be a continuous monotone exchangeable survival function and suppose that the two-dimensional marginal of $\bar{F}$ satisfies*

$$\forall t \in \Re_+,\ \bar{F}(t,0) = \bar{F}(2c(t/2,t/2)); \tag{10}$$

for some constant c (in fact $c \leq 1$ since $\bar{F}$ is a survival function), and

$$\bar{F} \sim \prod \mu; \tag{11}$$

for some continuous monotone univariate survival function μ; then $\bar{F}$ is l_p-isotropic with $p = -log_2(c)$.

Proof: By Theorem 1

$$\bar{F} = \int_0^\infty \prod \mu^s d\lambda_{\bar{F}}(s)$$

It suffices to show that $\mu(t) = K^{-t^p}$ for some constant K, and that $p = -\log_2(c)$. (10) and (11) entail that $\mu(t) = \mu(ct)^2$. After k iterations of the substitution $t \rightarrow ct$, this becomes:

$$\mu(t) = \mu(c^k t)^{2^k};$$

Put $t = c^x$; then μ must satisfy the functional equation:

$$\mu(c^x) = \mu(c^{k+x})^{2^k}$$

or

$$U(x) = k + U(k + x) \tag{12}$$

with $U(x) = \log_2 \ln \mu(c^x)$. As U is continuous, the solutions [1] have the form

$$U(x) = -x + \text{ constant}$$

so that

$$\mu(c^x) = e^{K2^{-x}} \tag{13}$$

for some constant K.

By theorem 9, all μ satisfying (11) are powers of a given solution, so we may choose $K = 1$. Substitute

$$x = log_c(t) = \frac{\log_2(t)}{\log_2(c)}$$

in (13): then in (13)

$$2^{-x} = t^{\frac{-1}{\log_2(c)}}.$$

It remains to show that $p = -\log_2(c)$. For an l_p -symmetric measure satisfying (11):

$$t = ||2c(t/2, t/2)||_p = c||(t,t)||_p = 2^{\frac{1}{p}}tc$$

which is equivalent to

$$c = 2^{-\frac{1}{p}}. \square$$

[1]This is an instance of Pexider's equation $f(x+y) = g(x) + h(y)$; having the general solution $f(t) = \phi(t) + a + b$, $g(t) = \phi(t) + a, h(t) = \phi(t) + b$; where a and b are arbitrary constants, and ϕ solves $\phi(x+y) = \phi(x) + \phi(y)$. [1, p. 142]

4 Functional equations for survival functions

Saying that two survival distributions are level entails functional equations from which the following two theorems draw conclusions. If a function f is continuous on $(0,1)$ and satisfies $f(xy) = f(x)f(y)$, then $f(x) = x^q$ for some constant q. ([1, p. 41]). For Theorems 9,10 the following simple fact will suffice:

Theorem 8 *Let f be a positive function defined on an interval $I \subset \Re_+$, such for all s, y, with $y, y^s \in I$, $f(y^s) = f(y)^s$; then $f(y) = y^q$, for some q.*

Proof: Pick $y_0 \in I, y_0 \neq 1$, and put $q = \log_{y_0} f(y_0)$. Then $f(y_0) = y_0^q$. Put $y = y_0^{r(y)}$ for $y \in I$. Then $f(y) = f(y_0^{r(y)}) = f(y_0)^{r(y)} = y_0^{qr(y)} = y^q$.□.

Theorem 9 *Let $\bar{F}$ and $\bar{G}$ be continuous monotone univariate survival functions, and suppose for some integer $n \geq 2$,*

$$\prod^n \bar{F} \sim \prod^n \bar{G} \tag{14}$$

then for some non-negative real number s;

$$\forall x \in \Re^n \; \bar{F}(x) = \bar{G}(x)^s. \tag{15}$$

Proof: Restricting to arguments of the type $\overbrace{(x_1, x_2, 0, 0\cdots)}^{n}$ it suffices to prove the result for $n = 2$. We have

$$\bar{F}(x_1)\bar{F}(x_2) = \bar{F}(t) \Longleftrightarrow \bar{G}(x_1)\bar{G}(x_2) = \bar{G}(t).$$

Put

$$\bar{F}(x) = \bar{H}(\bar{G}(x)),$$

which is possible because G is invertible. We have

$$\bar{H}\bar{G}(x_1)\bar{H}\bar{G}(x_2) = \bar{H}\bar{G}(t) = \bar{H}(\bar{G}(x_1)\bar{G}(x_2)). \tag{16}$$

This holds for all $\bar{G}(x_1)$, $\bar{G}(x_2) \in (0,1)$. With the multiplicative Cauchy equation [1, p. 41] we conclude for some $s \in \Re$

$$\bar{F}(x) = \bar{H}(\bar{G}(x)) = \bar{G}(x)^s.$$

Since $0 \leq \bar{F}(x), \bar{G}(x) \leq 1$, we must have $0 < s$.□

Theorem 10 *For univariate survival functions $\bar{F}_i$ and $\bar{G}$; with $\alpha_i > 0$, $i = 1, \cdots m$, $\sum \alpha_i = 1$; suppose*

$$\sum_{i=1}^{m} \alpha_i \prod \bar{F}_i \sim \prod \bar{G} \tag{17}$$

where $\sum_{i=1}^{m} \alpha_i \prod \bar{F}_i$, and $\bar{G}$ are continuous and strictly monotone. Then

$$\bar{F}_i = \bar{G}^{s_i} \tag{18}$$

Proof: The derivatives of $\bar{F}_i$ exist, though $\bar{F}_i$ are not assumed to be strictly monotone (this follows from the theorem). G^{-1} exists because of strict motonicity. Put $\mathcal{F}_i = \bar{F}_i\bar{G}^{-1}$. $\mathcal{F}_i$ are defined on $(0,1]$, and are continuous

($\bar{G}^{-1}(0)$ might be defined, but in light of continuity, there is no loss of generality in restricting attention to $(0,1]$.). Letting $m!$ denote the set of permutations of $\{1, \cdots m\}$, define:

$$
\begin{aligned}
d &= \min_{\pi \in m!}\{\inf\{x | \mathcal{F}_{\pi(1)}(y) \geq \mathcal{F}_{\pi(2)}(y) \geq \cdots \mathcal{F}_{\pi(m)}(y); \forall y \in (x,1)\}\} && (19)\\
D &= (d, 1] && (20)
\end{aligned}
$$

We note that $d < 1$ since the (one-sided) derivatives of $\mathcal{F}_i$ exist at 1. Also, if $d > 0$, and if S is the subset of indices for which $\mathcal{F}_i(d) = \mathcal{F}_j(d)$, then

$$
\text{there exist } y \in D \text{ and } i, j \in S \text{ such that } \mathcal{F}_i(y) \neq \mathcal{F}_j(y) \tag{21}
$$

We may assume without loss of generality that

$$
\mathcal{F}_1(y) \geq \mathcal{F}_2(y) \geq \cdots \mathcal{F}_m(y); \; y \in D. \tag{22}
$$

Suppose $(\prod \bar{G})(\overbrace{x, \cdots x}^{n}) = (\prod \bar{G})(\overbrace{t, \cdots t}^{q}, \overbrace{0, \cdots, 0}^{n-q})$, then $\bar{G}(x)^n = \bar{G}(t)^q$. We have from (17)

$$
\begin{aligned}
\bar{G}(x)^n &= \bar{G}(t)^q \\
& \Longrightarrow && (23)\\
\sum_{i=1}^{m} \alpha_i \bar{F}_i(x)^n &= \sum_{i=1}^{m} \alpha_i \bar{F}_i(t)^q
\end{aligned}
$$

Write $x = \bar{G}^{-1}(y)$; $t = \bar{G}^{-1}(v)$, and (23) is equivalent to:

$$\begin{aligned} y^n &= v^q \\ &\Longrightarrow \\ \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(y)^n &= \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(v)^q \end{aligned}$$

Substitute $v = y^{n/q}$. To study the limit behavior as $n \to \infty$ we restrict the arguments to D: For all y, n, q such that $y, y^{n/q} \in D$:

$$\left[\sum_{i=1}^{m} \alpha_i \mathcal{F}_i(y)^n\right]^{1/n} = \left[\sum_{i=1}^{m} \alpha_i \mathcal{F}_i(y^{n/q})^q\right]^{1/n} \tag{24}$$

We let $n, q \to \infty$, such that $n/q \to k$ with $y^k \in D$. The left hand side of (24) converges to $\mathcal{F}_1(y)$, for all $y, y^k \in D$, because of (22). On the right hand side

$$\begin{aligned} \left[\sum_{i=1}^{m} \alpha_i \mathcal{F}_i(y^{n/q})^q\right]^{1/n} &= \alpha_1^{1/n} \left[\mathcal{F}_1(y^{n/q})^q\right]^{1/n} \left[1 + \sum_{i=2}^{m} \frac{\alpha_i}{\alpha_1} \frac{\mathcal{F}_i(y^{n/q})^q}{\mathcal{F}_1(y^{n/q})^q}\right]^{1/n} \\ &\longrightarrow \mathcal{F}_1(y^k)^{1/k}. \end{aligned} \tag{25}$$

Comparing (24) and (25) we see that for $y, y^k \in D$; $\mathcal{F}_1(y^k) = \mathcal{F}_1(y)^k$. This means that Theorem 8 can be applied to yield; $\forall y \in D : \mathcal{F}_1(y) = y^{s_1}$; or:

$$\forall x \text{ such that } \bar{G}(x) \in D : \bar{F}_1(x) = \bar{G}(x)^{s_1}. \tag{26}$$

for some s_1, which must be positive since $\mathcal{F}_1$ ranges over $(0, 1]$. We now substitute (26) into (24). For $y, y^{n/q} \in D$ we can eliminate the term in $\mathcal{F}_1$ and conclude:

$$\sum_{i=2}^{m} \alpha_i \mathcal{F}_i(y)^n = \sum_{i=2}^{m} \alpha_i \mathcal{F}_i(y^{n/q})^q. \tag{27}$$

From this we derive, as above

$$\forall x \text{ such that } \bar{G}(x) \in D: \ \bar{F}_2(x) = \bar{G}(x)^{s_2}. \tag{28}$$

Proceeding in this way we show for $i = 1, \cdots m$

$$\forall x \text{ such that } \bar{G}(x) \in D: \ \bar{F}_i(x) = \bar{G}(x)^{s_i}. \tag{29}$$

We show that $d = 0$. Suppose to the contrary that $d > 0$. Then for those i, j such that $\mathcal{F}_i(d) = \mathcal{F}_j(d)$, we have by continuity: $\mathcal{F}_i(d) = d^{s_i}$ and $\mathcal{F}_j(d_j) = d^{s_j}$. This entails that $s_i = s_j$; and that $\mathcal{F}_i(y) = \mathcal{F}_j(y)$ for all $y \in D$. This however contradicts (21). It follows that D is $(0, 1]$, and the theorem is proved.□

In Theorem 10 *all* finite dimensional marginals are level, whereas in Theorem 9, the n-dimensional product measures are level, for a fixed n. It is not known whether the analogue of Theorem 9 holds for mixtures of product measures.

5 A de Finetti theorem

We recall briefly some facts about the topology of weak convergence. Let S be a metric space, and let $B(S)$ be the Borel field over S generated by the metric. The set $Z(S, B(S))$ of probability measures on $(S, B(S))$ may be endowed with the topology of weak convergence. For $\lambda \in Z(S, B(S))$, a basis neighborhood of λ may be written as:

$$N(\lambda) = \{\eta | \eta(F_i) \leq \lambda(F_i) + \xi; \xi > 0, F_i \in B(S) \text{ closed}; \; i = 1 \cdots k\} \tag{30}$$

Convergence in this topology is equivalent to weak convergence, denoted by $\rightarrow_w$. Moreover, the set of $\lambda \in Z(S, B(S))$ with finite support is dense in this topology on $Z(S, B(S))$. Let $\mathcal{M} = \{\mu | \mu$ is a univariate survival function$\}$. $\mathcal{M}$ is a metric space with the Prohorov metric, and may be endowed with the topology of weak convergence ([5, p. 236 - 238]). Define:

$$\Lambda = \text{ the set of probabilities on } (\mathcal{M}, B(\mathcal{M})).$$

Let $\mu^* \in \mathcal{M}$; define

$$\Lambda^* = \{\lambda \in \Lambda | \int_{\mathcal{M}} \prod \mu d\lambda\mu \sim \prod \mu^*\} \tag{31}$$

$$\Lambda_0^* = \{\lambda \in \Lambda^* | \lambda \text{ has finite support}\} \tag{32}$$

Let $\mu^* \in \mathcal{M}$; define If $\lambda \in \Lambda_0^*$; then $\lambda = \sum_{i=1}^n \alpha_i 1_{\mu_i}$, where 1 denotes the indicator function. We can now prove

Theorem1:. *Let $\bar{F}$: $[0, \infty)^\infty \rightarrow [0, 1]$ be a continuous monotone symmetric survival function and suppose that*

$$\bar{F} \sim \prod \mu^* \tag{33}$$

for some continuous monotone univariate survival function μ^**.Then* $\lambda_{\bar{F}}$ in (1) *is concentrated on* $\{\mu^{*s}|s \in (0,\infty)\}$.

Proof: Take Λ^* as above, with μ^* continuous monotone. Let $M = \{\mu^{*s}|s \in (0,\infty)\}$. We show that $\lambda_{\bar{F}}(M) = 1$ if $\lambda_{\bar{F}} \in \Lambda^*$. Pick $\lambda_i \rightarrow_w \lambda_{\bar{F}}$; $\lambda_i \in \Lambda_0^*, i = 1, 2 \cdots$; then $\lambda_i(M) = 1$ by Theorem 10. Let $\bar{M}$ denote the closure of M in $(\Lambda^*, B(\Lambda^*))$. We show:

$$\bar{M} = \{\mu^{*s}|s \in (0,\infty]\}$$

where $\mu^{*\infty}$ is unit mass at zero. Suppose $\mu^{*s_i}(x) \rightarrow_w \eta, i \rightarrow \infty$. We show that $\eta \in \{\mu^{*s}|s \in (0,\infty]\}$. Consider two cases: (1) for all $k \in N$ there are finitely many i with $s_i \in [0,k]$ and (2) there exists a $k \in N$ such that for infinitely many i, $s_i \in [0,k]$. In case (1) we have that $\mu^{*s_i}(x)$ converges to unit mass at the origin.

In case (2) pick a subsequence s_{i_j} such that $s_{i_j} \rightarrow r, r \in [0,k]$. For all $x \in [0,\infty)$

$$\mu^{*s_{i_j}}(x) \rightarrow \mu^{*r}(x).$$

It follows that $\mu^{*s_{i_j}} \rightarrow_w \mu^{*r}$; as $j \rightarrow \infty$, and hence that $\mu^{*s_i} \rightarrow_w \mu^{*r}$, since μ^{*s_i} converges weakly.

Now $\lambda_i(\bar{M}) = 1$ so that $\lambda_{\bar{F}}(\bar{M}) = 1$, by (30). Moreover, $\lambda_{\bar{F}}$ cannot assign positive mass to $\mu^{*\infty}$, as $\lambda_{\bar{F}}$ is continuous monotone by assumption.□

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